Zoom Class 18/4/2020

Ques For the curve in which $s = ce^{x/c}$ show that $c\rho = s\sqrt{s^2 - c^2}$

Soln Given $s = ce^{x/c}$ —— (1)

diffrn (1) w.r.t. x we get

$$\frac{ds}{dx} = \cancel{c}e^{x/c} \cdot \frac{1}{\cancel{c}}$$

$$\Rightarrow \frac{ds}{dx} = e^{x/c}$$

$$\Rightarrow \frac{1}{\cos\psi} = \frac{s}{c}$$

$\left\{ \because s = ce^{x/c} \Rightarrow e^{x/c} = s/c \text{ \& } dx/ds = \cos\psi \right\}$

$$\Rightarrow \sec\psi = \frac{s}{c} \text{ —— (A)}$$

$$\Rightarrow s = c\sec\psi \text{ —— (2)}$$

$\because$ Eqn (2) is in Intrinsic form

Diffrentiating it w.r.t ψ we get

$$\frac{ds}{d\psi} = c\sec\psi\tan\psi$$

$$\Rightarrow \rho = c\sec\psi\left(\sqrt{\sec^2\psi - 1}\right)$$

$\left\{ \because \tan^2\psi + 1 = \sec^2\psi \Rightarrow \tan\psi = \sqrt{\sec^2\psi - 1} \right\}$

SUNDA

$\Rightarrow \rho = \cancel{c}\,\frac{s}{\cancel{c}}\sqrt{\left(\frac{s}{c}\right)^2 - 1}$ — {using eqn Ⓐ}

$\Rightarrow \rho = s\sqrt{\frac{s^2 - c^2}{s^2}} \Rightarrow \rho = \frac{s}{c}\sqrt{s^2 - c^2}$

$\Rightarrow c\rho = s\sqrt{s^2 - c^2}$.

Proved

Ques Find the radius of curvature at any pt. of the parabola $y^2 = 4ax$ and deduce that radius of curvature at vertex of parabola is equal to latus rectum.

Soln $\because y^2 = 4ax$ is in Cartesian form we know that

$$\rho = \frac{\left(1 + \left(dy/dx\right)^2\right)^{3/2}}{\left(d^2y/dx^2\right)} \quad \text{—— ①}$$

Now $y^2 = 4ax \Rightarrow 2y\frac{dy}{dx} = 4a$ — {Diffrentiating w.r.t. x}

$\Rightarrow dy/dx = \frac{4a}{2y} = \frac{2a}{y}$ —— ②

Again Diffrentiating ② w.r.t. x we get

$\frac{d^2y}{dx^2} = \frac{-2a}{y^2}\frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{-2a}{y^2} \times \frac{2a}{y} = \frac{-4a^2}{y^3}$ — ③

∴ Substituting ② & ③ in ① we get

$$\cancel{\rho = \left(1-\left(\frac{2a}{y}\right)^2\right)^{3/2}}$$

$$\rho = \frac{\left(1+\left(\frac{2a}{y}\right)^2\right)^{3/2}}{\left(-\frac{4a^2}{y^3}\right)}$$

$$= -\frac{\left\{\frac{y^2+4a^2}{y^2}\right\}^{3/2}}{4a^2/y^3} = -\frac{(y^2+4a^2)^{3/2} \times y^3}{y^3 \times 4a^2}$$

$$\Rightarrow \rho = -\frac{(y^2+4a^2)^{3/2}}{4a^2}$$

$$\because y^2 = 4ax$$

05 THURSDAY
Week 10 ■ 065-301

$$\Rightarrow \rho = -\frac{\left\{(4ax)+4a^2\right\}^{3/2}}{4a^2}$$

$$= -\frac{(4a)^{3/2}(x+a)^{3/2}}{4a^2}$$

$$= -\frac{8a^{3/2}(x+a)^{3/2}}{4a^2}$$

$$= +2a^{-1/2}(x+a)^{3/2} \quad \text{——— ④}$$

Now, if radius of curvature is at vertex i.e., at pt. $(0,0)$ then $x=0$, $y=0$

then $\rho = +2a^{-1/2}(0+a)^{3/2} \leftarrow$ {sign is +ve bcoz of magnitude}

$$= 2a^{-1/2}\cdot a^{3/2} = 2a$$

$$\Rightarrow \rho = \frac{4a}{2} = \frac{1}{2}(4a) = \frac{1}{2}\text{(latus rectum)}.$$

Proved

Ques Find radius of curvature at point $(a\cos t, b\sin t)$ of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Soln Since coordinates of given point is in parametric form

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} \quad \text{—— (1)}$$

$\because x = a\cos t \Rightarrow x' = -a\sin t$

$\Rightarrow x'' = -a\cos t$

and

$y = b\sin t \Rightarrow y' = b\cos t$

$\Rightarrow y'' = -b\sin t$

SATURDAY

Week 10 067-299

07

Using the above values in Eqn (1) we get

$$\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{(-a\sin t)(-b\sin t) - (b\cos t)(-a\cos t)}$$

SUNDAY 08

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab\sin^2 t + ab\cos^2 t} = \frac{a^2\sin^2 t + b^2\cos^2 t}{ab(\sin^2 t + \cos^2 t)}$$

$$\Rightarrow \rho = \frac{(a\sin^2 t + b\cos^2 t)^{3/2}}{ab}$$

Ans

Ques ~~Prove~~ If ρ be radius of curvature of a parabola at a point whose distance along curve from fixed point is 's' then prove that

$$3\rho\frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{ds}\right)^2 - 9 = 0$$

Soln $\because$ Given curve is parabola therefore

$$y^2 = 4ax \Rightarrow y = \sqrt{4ax}$$

Diffrentiating w.r.t. 'x'

$$\Rightarrow \frac{dy}{dx} = \cancel{2}\sqrt{a}\cdot\frac{1}{\cancel{2}\sqrt{x}} = \sqrt{\frac{a}{x}} \quad \text{—— (1)}$$

Diffrentiating (1) again we get

10 TUESDAY Week 11 070-296

$$\frac{d^2y}{dx^2} = \sqrt{a}\left(-\frac{1}{2}\right)x^{-3/2}$$

$$= -\frac{1}{2}\frac{\sqrt{a}}{x^{3/2}} \quad \text{—— (2)}$$

then using (1) and (2) we get

$$\rho = \left(1+\left(\sqrt{\frac{a}{x}}\right)^2\right)^{3/2} \Big/ -\frac{1}{2}\frac{\sqrt{a}}{x^{3/2}}$$

$$= -\frac{(x+a)^{3/2}}{\cancel{x^{3/2}}} \times \frac{2\cancel{x^{3/2}}}{\sqrt{a}} = \frac{-2(x+a)^{3/2}}{\sqrt{a}}$$

In magnitude ρ is +ve

$$\therefore \rho = \frac{2(x+a)^{3/2}}{\sqrt{a}} \quad \text{—— (A)}$$

We know that

$$\frac{d\rho}{ds} = \frac{d\rho}{dx} \cdot \frac{dx}{ds} \quad \text{—— (3)}$$

and $$\frac{dx}{ds} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{—— (4)}$$

Now diffrentiating (A) w.r.t. x we get

$$\frac{d\rho}{dx} = \frac{\cancel{2}}{\sqrt{a}} \cdot \frac{3}{\cancel{2}}(x+a)^{1/2}$$

$$= \frac{3}{\sqrt{a}}\sqrt{x+a}.$$

& Eqn (1) & (4) $\Rightarrow$ $\frac{ds}{dx} = \sqrt{1+a/x} = \frac{\sqrt{x+a}}{\sqrt{x}}$

$$\therefore \frac{d\rho}{ds} = \frac{3}{\sqrt{a}}\sqrt{x+a} \cdot \frac{\sqrt{x}}{\sqrt{x+a}}$$

$$\frac{d\rho}{ds} = 3\sqrt{\frac{x}{a}} \quad \text{—— (5)}$$

Diffrentiating (5) w.r.t. 's' we get

$$\frac{d^2\rho}{ds^2} = \frac{d}{ds}\left(3\sqrt{\frac{x}{a}}\right)$$

$$= \frac{d}{dx}\left(3\sqrt{\frac{x}{a}}\right) \cdot \frac{dx}{ds} = \frac{3}{\sqrt{a}} \cdot \frac{1}{2\cancel{\sqrt{x}}} \times \frac{\cancel{\sqrt{x}}}{\sqrt{x+a}}$$

$$\frac{d^2\rho}{ds^2} = \frac{3}{2} \cdot \frac{1}{\sqrt{a}\sqrt{x+a}} \quad \text{—— (6)}$$

Substituting the values of (5) & (6) in LHS we get LHS $= 3\rho \frac{d^2\rho}{ds^2} - \left(\frac{d\rho}{dx}\right)^2 - 9$

$$= 3 \cdot \frac{\cancel{2}}{\sqrt{a}} (x+a)^{3/2} \cdot \frac{3}{\cancel{2}} \cdot \frac{1}{\sqrt{a} \cdot \sqrt{x+a}} - \frac{9x}{a} - 9$$

$$= 9 \cdot \frac{(x+a)^{3/2 - 1/2}}{a} - \frac{9x}{a} - 9$$

$$= \frac{9(x+a)}{a} - \frac{9x}{a} - 9$$

$$= \frac{9\cancel{x}}{\cancel{a}} + \frac{9\cancel{a}}{\cancel{a}} - \frac{9\cancel{x}}{\cancel{a}} - 9$$

$$= 9 - 9 = 0 = \text{RHS}$$

Proved